

Chapter 10

Differential Equations

Functions of one variable: $y, p, q, u, g, h, G, H, r, z$
Arguments (independent variables): x, y
Functions of two variables: $f(x, y), M(x, y), N(x, y)$
First order derivative: $y', u', \dot{y}, \frac{dy}{dt}, \dots$
Second order derivatives: $y'', \ddot{y}, \frac{d^2I}{dt^2}, \dots$
Partial derivatives: $\frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial x^2}, \dots$
Natural number: n
Particular solutions: y_1, y_p
Real numbers: $k, t, C, C_1, C_2, p, q, \alpha, \beta$
Roots of the characteristic equations: λ_1, λ_2
Time: t
Temperature: T, S
Population function: $P(t)$
Mass of an object: m
Stiffness of a spring: k
Displacement of the mass from equilibrium: y
Amplitude of the displacement: A
Frequency: ω
Damping coefficient: γ
Phase angle of the displacement: δ
Angular displacement: θ
Pendulum length: L

Acceleration of gravity: g
 Current: I
 Resistance: R
 Inductance: L
 Capacitance: C

10.1 First Order Ordinary Differential Equations

1164. Linear Equations

$$\frac{dy}{dx} + p(x)y = q(x).$$

The general solution is

$$y = \frac{\int u(x)q(x)dx + C}{u(x)},$$

where

$$u(x) = \exp\left(\int p(x)dx\right).$$

1165. Separable Equations

$$\frac{dy}{dx} = f(x, y) = g(x)h(y)$$

The general solution is given by

$$\int \frac{dy}{h(y)} = \int g(x)dx + C,$$

or

$$H(y) = G(x) + C.$$

1166. Homogeneous Equations

The differential equation $\frac{dy}{dx} = f(x, y)$ is homogeneous, if the function $f(x, y)$ is homogeneous, that is $f(tx, ty) = f(x, y)$.

The substitution $z = \frac{y}{x}$ (then $y = zx$) leads to the separable equation

$$x \frac{dz}{dx} + z = f(1, z).$$

1167. Bernoulli Equation

$$\frac{dy}{dx} + p(x)y = q(x)y^n.$$

The substitution $z = y^{1-n}$ leads to the linear equation

$$\frac{dz}{dx} + (1-n)p(x)z = (1-n)q(x).$$

1168. Riccati Equation

$$\frac{dy}{dx} = p(x) + q(x)y + r(x)y^2$$

If a particular solution y_1 is known, then the general solution can be obtained with the help of substitution

$z = \frac{1}{y - y_1}$, which leads to the first order linear equation

$$\frac{dz}{dx} = -[q(x) + 2y_1 r(x)]z - r(x).$$

1169. Exact and Nonexact Equations

The equation

$$M(x, y)dx + N(x, y)dy = 0$$

is called **exact** if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},$$

and **nonexact** otherwise.

The general solution is

$$\int M(x, y)dx + \int N(x, y)dy = C.$$

1170. Radioactive Decay

$$\frac{dy}{dt} = -ky,$$

where $y(t)$ is the amount of radioactive element at time t , k is the rate of decay.

The solution is

$$y(t) = y_0 e^{-kt}, \text{ where } y_0 = y(0) \text{ is the initial amount.}$$

1171. Newton's Law of Cooling

$$\frac{dT}{dt} = -k(T - S),$$

where $T(t)$ is the temperature of an object at time t , S is the temperature of the surrounding environment, k is a positive constant.

The solution is

$$T(t) = S + (T_0 - S)e^{-kt},$$

where $T_0 = T(0)$ is the initial temperature of the object at time $t = 0$.

1172. Population Dynamics (Logistic Model)

$$\frac{dP}{dt} = kP \left(1 - \frac{P}{M} \right),$$

where $P(t)$ is population at time t , k is a positive constant, M is a limiting size for the population.

The solution of the differential equation is

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-kt}}, \text{ where } P_0 = P(0) \text{ is the initial population at time } t = 0.$$

10.2 Second Order Ordinary Differential Equations

1173. Homogeneous Linear Equations with Constant Coefficients

$$y'' + py' + qy = 0.$$

The characteristic equation is

$$\lambda^2 + p\lambda + q = 0.$$

If λ_1 and λ_2 are distinct real roots of the characteristic equation, then the general solution is

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}, \text{ where}$$

C_1 and C_2 are integration constants.

If $\lambda_1 = \lambda_2 = -\frac{p}{2}$, then the general solution is

$$y = (C_1 + C_2 x) e^{-\frac{p}{2}x}.$$

If λ_1 and λ_2 are complex numbers:

$\lambda_1 = \alpha + \beta i$, $\lambda_2 = \alpha - \beta i$, where

$$\alpha = -\frac{p}{2}, \quad \beta = \frac{\sqrt{4q - p^2}}{2},$$

then the general solution is

$$y = e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x).$$

1174. Inhomogeneous Linear Equations with Constant Coefficients

$$y'' + py' + qy = f(x).$$

The general solution is given by

$$y = y_p + y_h, \text{ where}$$

y_p is a particular solution of the inhomogeneous equation and y_h is the general solution of the associated homogeneous equation (see the previous topic 1173).

If the right side has the form

$$f(x) = e^{\alpha x} (P_1(x) \cos \beta x + P_2(x) \sin \beta x),$$

then the particular solution y_p is given by

$$y_p = x^k e^{\alpha x} (R_1(x) \cos \beta x + R_2(x) \sin \beta x),$$

where the polynomials $R_1(x)$ and $R_2(x)$ have to be found by using the [method of undetermined coefficients](#).

- If $\alpha + \beta i$ is not a root of the characteristic equation, then the power $k = 0$,
- If $\alpha + \beta i$ is a simple root, then $k = 1$,
- If $\alpha + \beta i$ is a double root, then $k = 2$.

1175. Differential Equations with y Missing

$$y'' = f(x, y').$$

Set $u = y'$. Then the new equation satisfied by v is

$$u' = f(x, u),$$

which is a first order differential equation.

1176. Differential Equations with x Missing

$$y'' = f(y, y').$$

Set $u = y'$. Since

$$y'' = \frac{du}{dx} = \frac{du}{dy} \frac{dy}{dx} = u \frac{du}{dy},$$

we have

$$u \frac{du}{dy} = f(y, u),$$

which is a first order differential equation.

1177. Free Undamped Vibrations

The motion of a Mass on a Spring is described by the equation

$$m\ddot{y} + ky = 0,$$

where

m is the mass of the object,

k is the stiffness of the spring,

y is displacement of the mass from equilibrium.

The general solution is

$$y = A \cos(\omega_0 t - \delta),$$

where

A is the amplitude of the displacement,

ω_0 is the fundamental frequency, the period is $T = \frac{2\pi}{\omega_0}$,

δ is phase angle of the displacement.

This is an example of simple harmonic motion.

1178. Free Damped Vibrations

$$m\ddot{y} + \gamma\dot{y} + ky = 0, \text{ where}$$

γ is the damping coefficient.

There are 3 cases for the general solution:

Case 1. $\gamma^2 > 4km$ (overdamped)

$$y(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t},$$

where

$$\lambda_1 = \frac{-\gamma - \sqrt{\gamma^2 - 4km}}{2m}, \quad \lambda_2 = \frac{-\gamma + \sqrt{\gamma^2 - 4km}}{2m}.$$

Case 2. $\gamma^2 = 4km$ (critically damped)

$$y(t) = (A + Bt)e^{\lambda t},$$

where

$$\lambda = -\frac{\gamma}{2m}.$$

Case 3. $\gamma^2 < 4km$ (underdamped)

$$y(t) = e^{-\frac{\gamma}{2m}t} A \cos(\omega t - \delta), \text{ where}$$

$$\omega = \sqrt{4km - \gamma^2}.$$

1179. Simple Pendulum

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0,$$

where θ is the angular displacement, L is the pendulum length, g is the acceleration of gravity.

The general solution for small angles θ is

$$\theta(t) = \theta_{\max} \sin \sqrt{\frac{g}{L}}t, \text{ the period is } T = 2\pi \sqrt{\frac{L}{g}}.$$

1180. RLC Circuit

$$L \frac{d^2I}{dt^2} + R \frac{dI}{dt} + \frac{1}{C}I = V'(t) = \omega E_0 \cos(\omega t),$$

where I is the current in an RLC circuit with an ac voltage source $V(t) = E_0 \sin(\omega t)$.

The general solution is

$$I(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} + A \sin(\omega t - \varphi),$$

where

$$r_{1,2} = \frac{-R \pm \sqrt{R^2 - \frac{4L}{C}}}{2L},$$

$$A = \frac{\omega E_0}{\sqrt{\left(L\omega^2 - \frac{1}{C}\right)^2 + R^2\omega^2}},$$

$$\varphi = \arctan\left(\frac{L\omega}{R} - \frac{1}{RC\omega}\right),$$

C_1, C_2 are constants depending on initial conditions.

10.3. Some Partial Differential Equations

1181. The Laplace Equation

$$\frac{\partial^2 \mathbf{u}}{\partial x^2} + \frac{\partial^2 \mathbf{u}}{\partial y^2} = 0$$

applies to potential energy function $u(x, y)$ for a conservative force field in the xy -plane. Partial differential equations of this type are called **elliptic**.

1182. The Heat Equation

$$\frac{\partial^2 \mathbf{u}}{\partial x^2} + \frac{\partial^2 \mathbf{u}}{\partial y^2} = \frac{\partial \mathbf{u}}{\partial t}$$

applies to the temperature distribution $u(x,y)$ in the xy -plane when heat is allowed to flow from warm areas to cool ones. The equations of this type are called **parabolic**.

1183. The Wave Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial t^2}$$

applies to the displacement $u(x,y)$ of vibrating membranes and other wave functions. The equations of this type are called **hyperbolic**.